‘Black Box’ Circuit Analysis

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Abstract

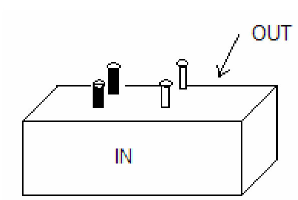
The purpose of this experiment was to identify the circuit elements present in black boxes four and five. Through a series of tests, the following conclusions were obtained:

1) Box four contained a 1.06kΩ resistor in parallel with a 0.172μF capacitor; the combination was in series with a 1.97kΩ resistor. Theoretically, this would translate to a 1kΩ resistor in parallel with a 0.2μF, in series with a 2kΩ resistor

2) Box five contained a 1.02kΩ resistor in series with a 2.1kΩ resistor, equivalent to a resistance of 3.12 kΩ. This would translate to a theoretical circuit with a 1kΩ resistor in series with a 2kΩ resistor.

Procedure

Choosing a 'black box' among five others, the group then conducted tests to determine the arrangement of the electronic components inside without opening the box. The hint was given that each box contained *at most* two resistors, one inductor, one capacitor, and one op amp, and that no box contained more than four of the components in this list while most contained two or three elements. The same testing procedure was repeated for a second box.



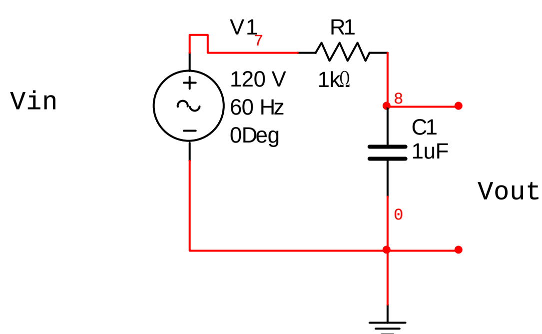
Simplified diagram of 'black box' with leads

Theory

Determining the circuits in the boxes relied on an understanding of time-domain response, frequency response, filters, and the properties of inductors, capacitors, and resistors. This becomes apparent when reviewing the process used to determine box four.

The printout from the dynamic signal analyzer is shown below. One can compare the printout to the magnitude portion of a bode plot (frequency response) given by the transfer function. In short, the frequency response describes how an output is related to an input over varying frequencies. The transient response relates the output to the input over time.

The printout for box five produced a straight line. Knowledge of how to calculate the frequency response or transfer function is essential to the interpretation of the printouts.



To find the frequency (ω) response of the above circuit one can use voltage divider. This requires using the impedance of the capacitor, as voltage divider gives $Vout = \frac{Z_c}{Z_c + R} Vin$.

Z_c can be defined as $\frac{1}{sC}$, where s = jω for a sinusoidal input. To prove this, one remembers that because the source is AC, it can be written as using complex numbers and $V = V_m e^{j\omega t}$. Now remember $I_c = C\frac{dv}{dt}$. This means $\frac{dv}{dt} = j\omega V_m e^{j\omega t}$. Then $I_c = \frac{I}{Cj\omega} = V$, which using phasor form can be written as I=$Cj\omega V$. Using ohms law and impedance (V=IZ), this equation is manipulated to $\frac{I}{Cj\omega} = V$. Therefore $Z_c = \frac{1}{j\omega C}$. S can be seen to be $j\omega$.

Replacing the derived Z_c into $Vout = \frac{Z_c}{Z_c + R} Vin$., yields $Vout = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} Vin$.

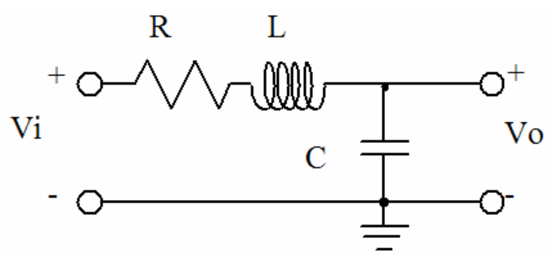
Simplifying: $Vout = \frac{\frac{1}{j\omega C}}{\frac{1 + Rj\omega C}{j\omega C}} Vin$, then $Vout = \frac{1}{j\omega C} \bullet \frac{j\omega C}{1 + Rj\omega C} Vin = \frac{1}{1 + Rj\omega C} Vin$.

Attenuation is the magnitude of $\frac{Vout}{Vin}$, or $\left|\frac{Vout}{Vin}\right| = \frac{1}{\sqrt{1 + (R\omega C)^2}}$. Replacing jω with s yields the transfer function: $\frac{Vout}{Vin} = \frac{1}{1 + sRC}$. The transfer function thus clearly relates the output to the frequency of the input. To calculate the magnitude Bode plot one simply converts the transfer function into phasor form, and plots the magnitude in decibels.

This knowledge allows one to easily predict what is in box number five. The horizontal line dictates that the circuit is purely resistive, as the magnitude is constant for all frequencies. Although a circuit with reactive elements can appear purely resistive for a given frequency, a circuit must be purely resistive to yield a constant magnitude for the transfer function for all frequencies.

The transient or time response is also helpful in determining the circuit elements in black boxes. Again, it is helpful to review the theory of the transient response. In an attempt to gain an understanding of the transient response of second order circuits, students must become familiar with the important characteristics of passive elements and their impact on circuit behavior. One solves for the transient response for voltage or current by first using Kirchoff's current or voltage laws and obtaining a differential equation that yields a characteristic equation, which in turn gives the homogeneous portion (natural response) of the transient response. Impedance can also be used to calculate the homogeneous response. To find the complete transient response, one must calculate the particular response, the source dependent response of a circuit. This is done by determining what the output is at steady-state, or as time approaches infinity. The form of the particular response is the form of the input and all of its derivatives. Combining the homogeneous and particular responses and solving with initial conditions gives the complete response of the circuit. With this function one can determine the relationship of the input to desired output over any time interval.

An example of calculating transient response is as follows (from Lab 4):



Using impedance, we are trying to find an expression for the voltage across the capacitor (v_c). So, we create terminals across the capacitor C as shown above. Noting that C (impedance, $Z = \frac{1}{sC}$) is in parallel with the inductor, L (Z = sL) which is in series with the resistor R (Z = R), we have

$$Z_{ab}(s) = \frac{(R+sL)\left(\frac{1}{sC}\right)}{R+SL+\frac{1}{sC}}$$

Simplifying,

$$Z_{ab}(s) = \frac{(R+sL)}{\left(R+sL+\frac{1}{sC}\right)sC} = \frac{R+sL}{sCR+s^2LC+1} = \frac{L}{LC}\frac{s+\frac{R}{L}}{s^2+s\frac{R}{L}+\frac{1}{LC}}$$

Since $Z_{ab} = \infty$ as it is an open circuit, we send the denominator of Z_{ab}(s) to zero yielding the characteristic equation:

$$s^2 + s\frac{R}{2L} + \frac{1}{LC} = 0\,.$$

Using the quadratic equation to solve, we have

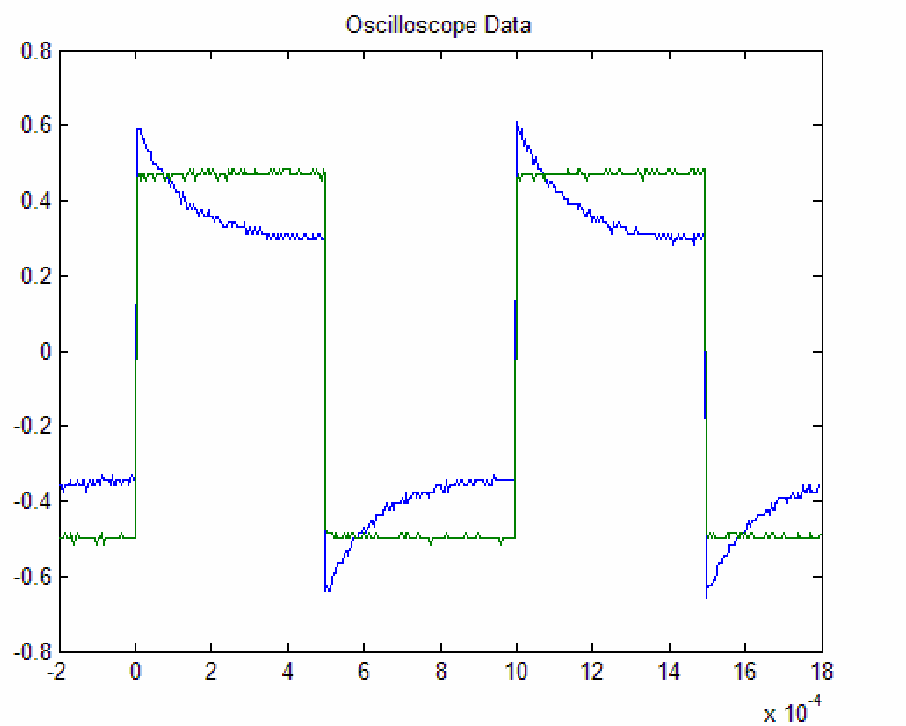
$$s = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

If R=3 and L & C =1, s=-.38, -1.11. The homogeneous response is therefore over damped (2 real roots).

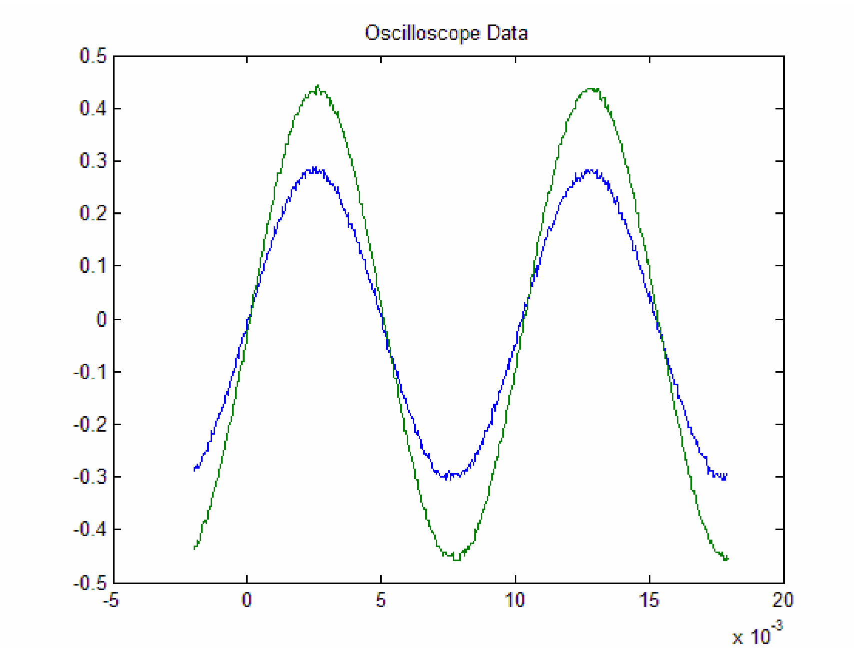
The output voltage as t approaches infinity is V_{in} (assuming step input) as the capacitor acts as an open circuit (no current through resistor or inductor). The complete response can be written as: $Vout = Vin + Ae^{-.38t} + Be^{-1.19t}$.

By analyzing the damping of the circuit in the black box and using the characteristics of homogeneous response, one can gain insight to the components of that circuit.

Experimental tests on box 4

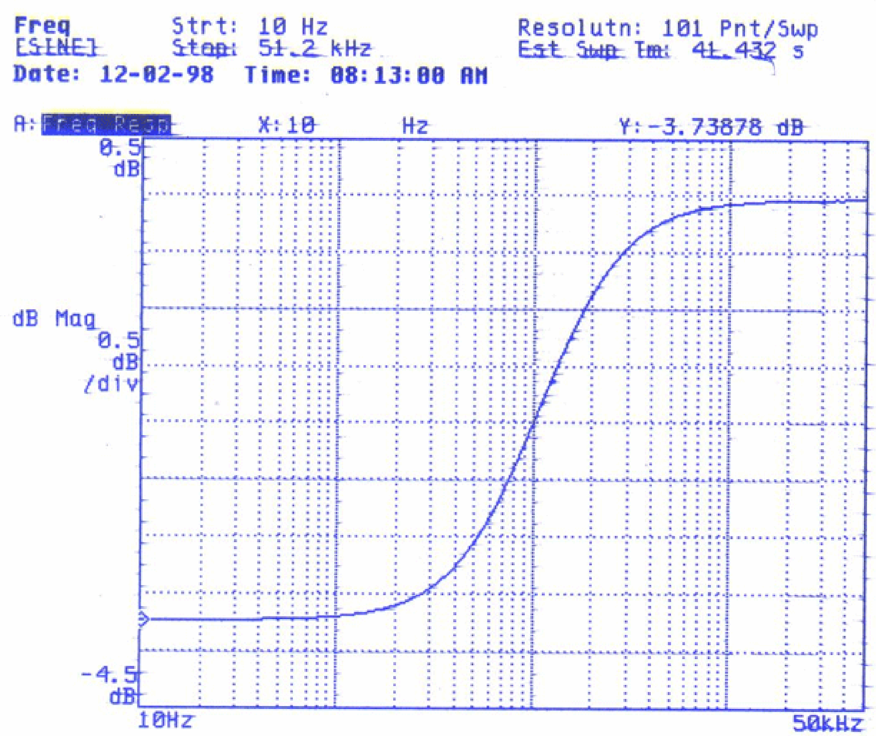


The transient response of the circuit using a square wave input is reminiscent of the continual charging and discharging of a capacitor due to the input. From this it was quickly determined that either **an inductor or a capacitor** was inside the circuit box.



There is a slight phase shift in this sinusoidal input plot, implying that there must be reactive elements.

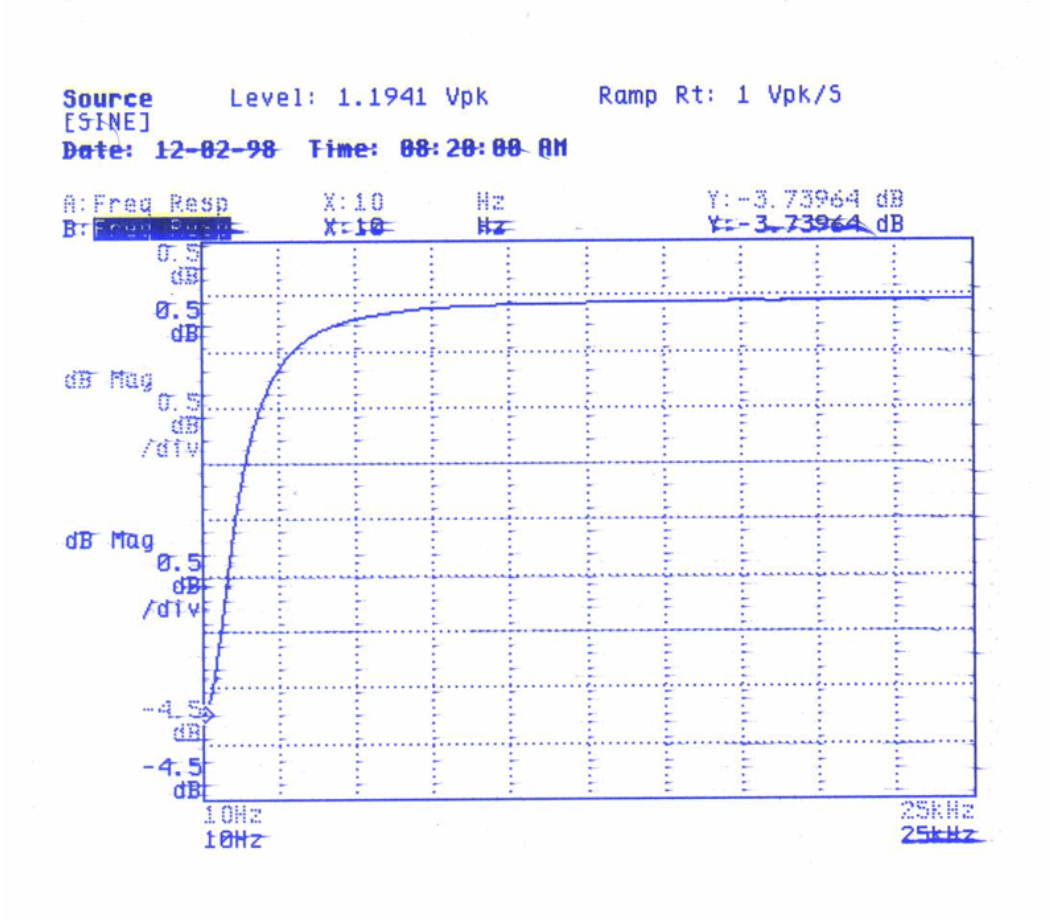
With an input of 97Hz and V_{in} as 900mV, we obtained an output voltage of 600mV.



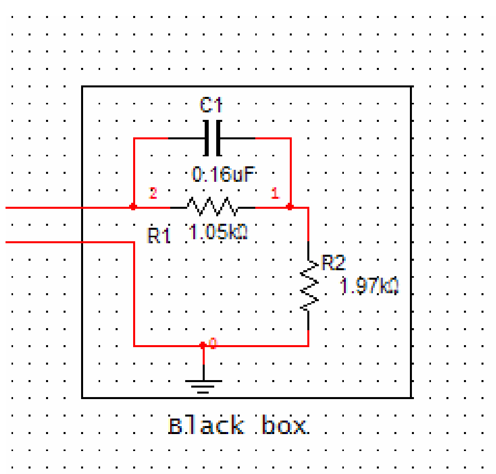
From the same output printed from the digital signal analyzer on a log scale it is clear that the circuit is **high pass** as it attenuates low frequencies but passes higher frequencies. The available options for first order high-pass circuits include an RC circuit (with the output measured across the resistor) or an RL circuit (with the output measured across the inductor). In order to distinguish between the two, the box was connected to an RLC meter. The measurements obtained showed a circuit containing a capacitor and resistors.

RLC meter measurements

	Values	**Box 4**	**Box 5**
Measured across both non-grounded leads	Resistance	1.05kΩ	1.02kΩ
	Inductance	0	0
	Capacitance	0.16μF	0
Measured across both output leads	Resistance	1.97kΩ	2.1kΩ
	Inductance	0	0
	Capacitance	0	0



This graph substantiates the claim that the circuit is first order because no oscillation or peak is present. The hypothesis of the contents of the black box number four is as follows:



The transfer function for box four is as follows:

$$V_0 = \frac{Z_{R2}}{(Z_C \parallel Z_{R1}) + Z_{R2}}$$

$$\text{H(s)} = \frac{V_0}{V_i} = \frac{R_2}{\dfrac{R_1}{R_1 sC + 1} + R_2} = \frac{R_2}{\left(\dfrac{R_2(R_1 sC + 1) + R_1}{R_1 sC + 1}\right)} = \frac{R_2 R_1 sC + R_2}{R_2 R_1 sC + (R_2 + R_1)}$$

At low frequencies, we approximate s = jω → 0:

$$\text{H(0)} = \frac{R_2}{R_2 + R_1} = \frac{2}{3}$$

In decibels,

$$20 \log_{10}\left(\frac{2}{3}\right) = -3.52$$

Compared to the output, this is an error of 6.25%. This is expected, because the resistor values are not as accurate as their theoretical values.

To find the cutoff frequency, we calculate H(jω):

$$\text{H(j}\omega) = \frac{R_2 + (R_2 R_1 \omega C) j}{(R_2 + R_1) + (R_2 R_1 \omega C) j}$$

The magnitude of which is:

$$\frac{\sqrt{R_2{}^2 + (R_2 R_1 \omega C)^2}}{\sqrt{(R_2 + R_1)^2 + (R_2 R_1 \omega C)^2}}$$; at the cutoff frequency, this is equal to $\frac{1}{\sqrt{2}}$

Solving by squaring both sides:

$$\frac{R_2{}^2 + (R_2 R_1 \omega C)^2}{(R_2 + R_1)^2 + (R_2 R_1 \omega C)^2} = \frac{1}{2}$$

$$2R_2{}^2 + 2(R_2{}^2 R^2{}_1 \omega^2 C^2) = (R_2{}^2 + R_1{}^2 + 2R_1 R_2) + (R_2{}^2 R^2{}_1 \omega^2 C^2)$$

$$\omega_c{}^2 = \frac{(-R_2{}^2 + R_1{}^2 + 2R_1 R_2)}{(R_2{}^2 R^2{}_1 C^2)}$$

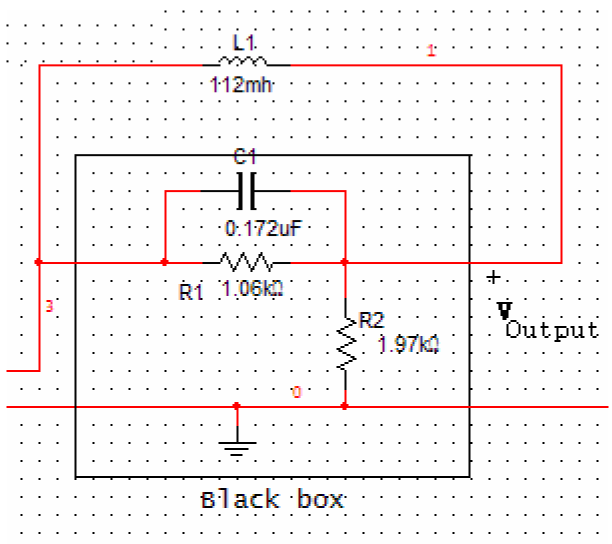
$$\omega_c = \sqrt{\frac{(-R_2{}^2 + R_1{}^2 + 2R_1R_2)}{(R_2{}^2R_1{}^2C^2)}} = \sqrt{\frac{(-2000^2 + 1000^2 + 2(2000000)}{(2000^2 \cdot 1000^2 \cdot 0.2 \cdot 10^{-6})}} = 7500$$

$$f = \frac{\omega_c}{2\pi} = 1193.7\text{Hz}$$

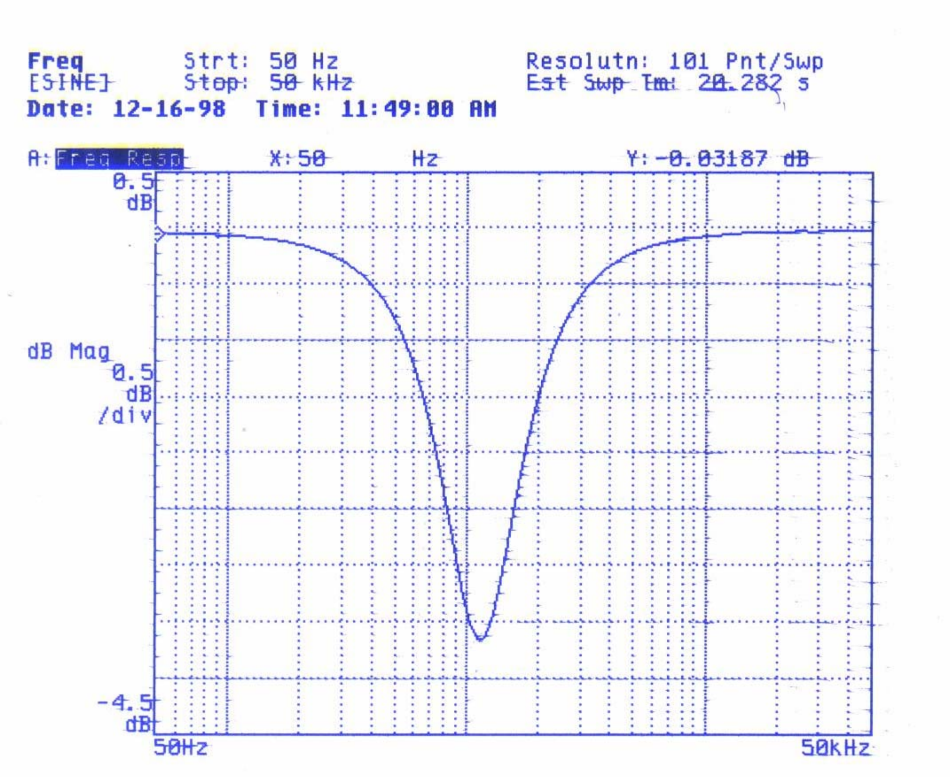
This value is reflected in the output from the digital signal analyzer, reinforcing the claim.

Further analyses

To further complement the analysis of the box as well as the hypothesis, an inductor was connected in parallel to the capacitor and resistors in our working hypothesis, as shown below.



The following graph was obtained:



The graph is a band reject filter, as low and high frequencies are passed, but a particular width of frequencies is attenuated.

To understand why, one constructs the transfer function for the circuit.

From voltage divider: $V_{out} = V_{in}\left(\frac{R_2}{R_2 + Z_{R_1LC}}\right)$

The transfer function is thus $\frac{V_{out}}{V_{in}} = \frac{R_2}{R_2 + Z_{R_1LC}}$.

Now one must calculate the impedance of Z_{RLC}:

$$\frac{1}{Z_{R_1LC}} = \frac{1}{sL} + sC + \frac{1}{R_1}$$

Simplifying: $\frac{1}{Z_{R_1LC}} = \frac{R_1 + s^2CLR_1 + sL}{sLR_1}$

$$Z = \frac{sLR_1}{R_1 + s^2CLR_1 + sL}$$

Plugging this value in for the transfer function yields:

$$\frac{V_{out}}{V_{in}} = \frac{R_2}{\dfrac{sLR_1}{R_1 + s^2CLR_1 + sL} + R_2}$$

Simplifying this equation:

$$\frac{V_{out}}{V_{in}} = \frac{R_2R_1 + s^2CLR_1R_2 + sLR_2}{sLR_1 + R_2R_1 + s^2CLR_1R_2 + sLR_2}$$

Finally one has: $$\frac{V_{out}}{V_{in}} = \frac{s^2 + s\left(\dfrac{1}{CR_1}\right) + \dfrac{1}{LC}}{s^2 + s\left(\dfrac{1}{CR_2} + \dfrac{1}{CR_1}\right) + \dfrac{1}{LC}}$$

Now one analyzes

A.) As $\omega \to \infty$ (knowing that $s = j\omega$), leaves $\frac{V_{out}}{V_{in}} = \frac{1}{1}$, or zero in decibels .

B.) As $\omega \to 0$, leaves $\frac{V_{out}}{V_{in}} = \frac{1}{1}$, or zero in decibels.

Therefore it appears as a band-reject filter. To check, one looks at ω_o, which equals $\frac{1}{\sqrt{LC}}$. Plugging the values we predicted:

$$\omega_o = \frac{1}{\sqrt{\left(112 \cdot 10^{-3}\right)\left(0.2 \cdot 10^{-6}\right)}} = 6681.5$$

$$f = \frac{\omega}{2\pi} = 1063.7\text{Hz}$$

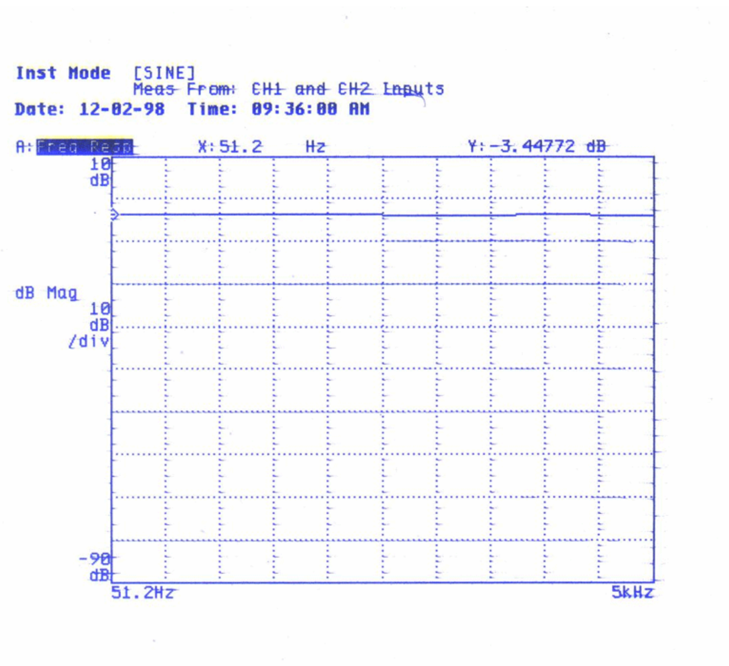
This is very close to the value found in the graph. At this frequency, the filter should be at a maximum magnitude because this is the frequency where the impedances from the capacitor and inductor are of equal and opposite value, therefore yielding a purely resistive circuit. However, because this frequency is attenuated or not of the pass band, it is a minimum. Plugging in this frequency into the transfer function gives a magnitude of $\frac{R_2}{R_1 + R_2}$.

At resonance, the magnitudes of the impedances are equal but the vectors are 180 degrees out of phase with each other and thus cancel.[1] At resonance, therefore, the resistor dominates.

Analysis of Box 5

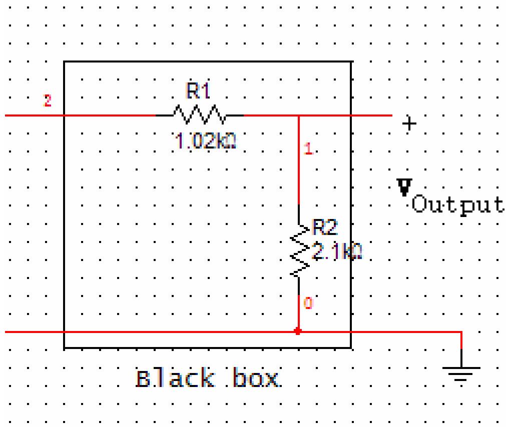
Due to the extra weight resulting from the batteries for the operational amplifiers, it was clear that only boxes one and six contained op amps. The analysis was therefore restricted to inductors, capacitors, and resistors as circuit elements.

From the digital signal analyzer printout of frequency vs. magnitude on a linear scale, it is clear that box five is a first order circuit as there is no oscillation in the curve:



The analysis was completed by the use an ohmmeter to measure two resistor values. It is clear that the resistor values obtained are *equivalent* resistor values- there could be resistors in series or in parallel across the leads that will result in the same equivalent resistance measured by the ohmmeter.

[1] “totse.com | A Different Point of View” http://www.totse.com/en/fringe/gravity_anti_gravity/gravres.html



The above diagram illustrates the contents of the box as experimentally determined in this lab.

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